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Equilogical spaces and algebras for a double-power monad

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Abstract

We investigate the algebras for the double-power monad on the Sierpisnki space in the category $\mathcal{E}qu$ of equilogical spaces, a cartesian closed extension of $\mathcal{T}op_0$ introduced by Scott, and the relationship of such algebras with frames. In particular, we focus our attention on interesting subcategories of $\mathcal{E}qu$. We prove uniqueness of the algebraic structure for a large class of equilogical spaces, and we characterize the algebras for the double-power monad in the category of algebraic lattices and in the category of continuous lattices, seen as full subcategories of $\mathcal{E}qu$.

We also analyse the case of algebras in the category $\mathcal{T}op_0$ of T_0-spaces, again seen as a full subcategoy of $\mathcal{E}qu$, proving that each algebra for the double-power monad in $\mathcal{T}op_0$ has an underlying sober, compact, connected space.

1 Introduction

The category $\mathcal{E}qu$ of equilogical spaces, introduced by Scott in [Sco96], provides a useful extension of the category $\mathcal{T}op_0$ of T_0-spaces; indeed, $\mathcal{E}qu$ is locally cartesian closed, see [CR00], and the inclusion functor $\mathcal{T}op_0 \hookrightarrow \mathcal{E}qu$ preserves the (existing) cartesian closed structure.

Considering the Sierpinski space Σ as an equilogical space, the self-adjoint functor $\Sigma^{(-)}\colon \mathcal{E}qu \to \mathcal{E}qu^{\mathrm{op}}$ gives rise to a monad on $\mathcal{E}qu$: the double-power monad on Σ, which we denote $\mathbf{\Sigma^2}$ in the following. Monads of this kind have been widely studied in different contexts: Taylor developed Abstract Stone Duality investigating certain double-power monads under the weak assumption that in category $\mathcal{C}$ the object Σ can be exponentiated, see [Tay02a, Tay02b]. Townsend and Vickers analysed the double-power monad for Σ seen as a locale—aka the frame of open subsets of Σ—and they compared it to other important constructions in the category of locales, see [VT04, Vic04]. Such monads played an important role also in the abstract approach to semantics of computations, see *e.g.* [PR97, PT97, Thi97b, Füh99] and connections with the computational idea of continuations, explored in particular in Hayo Thielecke's PhD thesis [Thi97a], certainly require further consideration.

In investigating the double-power monad on the Sierpinski topological space in a cartesian closed extension of the category of T_0-spaces, one must take into account how fundamental a role the functor $\Sigma^{(-)}$ plays in the study of exponentiability in $\mathcal{T}op$, as it is well-known that a topological space X is exponentiable if and only if the exponential Σ^X exists in $\mathcal{T}op$. The starting point of our investigation is a connection between $\mathbf{\Sigma^2}$-algebras in $\mathcal{E}qu$ and frames; we know from [Dub70] that, since $\mathcal{E}qu$ is cartesian closed, the algebras for the double-power monad in $\mathcal{E}qu$ are the internal models of the algebraic theory of Σ in $\mathcal{E}qu$. As a consequence of this, in [FRS17] it was shown that the $\mathbf{\Sigma^2}$-algebra structure on an equilogical space E induces a frame structure on the global sections of E. On the other hand, the Lawvere finitary algebraic theory in $\mathcal{E}qu$ of Σ is precisely that of internal frames in $\mathcal{E}qu$. The basic question we address in the present paper is whether the

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internal theory of Σ coincide with its finitary part. In principle, one would expect a positive answer since frames appear as the "algebrization" of open subsets, but to get such a positive answer has proved extremely difficult. After all the work we put into the question, we suspect that the answer is more subtle and we conjecture that the $\mathbf{\Sigma^2}$-algebras in $\mathcal{Equ}$ are only particular frames.

The study in the present paper focuses on certain subcategories of $\mathcal{Equ}$ on which one can restrict the double-power monad in order to analyse the $\mathbf{\Sigma^2}$-algebras in these subcategories. These include the category $\mathcal{AlgLatt}$ of algebraic lattices and the category $\mathcal{ContLatt}$ of continuous lattices, together with two other important subcategories, $\mathcal{REqu}$ and $\mathcal{SEqu}$, introduced in section 2, which are very relevant for the structure of the category $\mathcal{Equ}$.

In section 3, after briefly recalling known facts about the double-power monad $\mathbf{\Sigma^2}$ and its algebras, we characterize $\mathcal{ContLatt}$ as the intersection of the two subcategories $\mathcal{SEqu}$ and $\mathcal{REqu}$.

In section 4 we outline some properties of the frame structure of a $\mathbf{\Sigma^2}$-algebra in $\mathcal{AlgLatt}$ and in $\mathcal{REqu}$, and we characterize $\mathbf{\Sigma^2}$-homomorphisms between them. We also prove the uniqueness of the structure map for $\mathbf{\Sigma^2}$-algebras in $\mathcal{AlgLatt}$, in $\mathcal{REqu}$ and in $\mathcal{ContLatt}$.

In section 5, we apply the previous results to obtain a characterization for $\mathbf{\Sigma^2}$-algebras in $\mathcal{AlgLatt}$ and $\mathcal{ContLatt}$, involving algebraic frames and continuous frames.

Finally, in section 6, we investigate the case of $\mathbf{\Sigma^2}$-algebras in $\mathcal{Top}_0$, analysing the topological space $\Sigma^{(\Sigma^X)}$ for a T_0-space X. We show that every $\mathbf{\Sigma^2}$-algebra in $\mathcal{Top}_0$ is a sober, compact and connected space, but not necessarily locally compact.

2 Preliminaries

Recall from [BBS04, Sco96] that an ***equilogical space*** is a triple $E = (|E|, \tau_E, \equiv_E)$, where $(|E|, \tau_E)$ is a T_0-space and $\equiv_E$ is an equivalence relation on $|E|$. Given equilogical spaces $(|E|, \tau_E, \equiv_E)$ and $(|F|, \tau_F, \equiv_F)$, we say that a continuous function $f\colon (|E|, \tau_E) \to (|F|, \tau_F)$ is ***equivariant*** if, for every $x, x' \in |E|$ such that $x \equiv_E x'$, one has that $f(x) \equiv_F f(x')$. Two continuous equivariant functions $f, f'\colon (|E|, \tau_E) \to (|F|, \tau_F)$ are ***equivalent***—and we shall write $f \equiv_{F^E} f'$—if for all $x \in |E|$, one has that $f(x) \equiv_F f'(x)$.

The category $\mathcal{Equ}$ of equilogical spaces consists of

objects are the equilogical spaces $E = (|E|, \tau_E, \equiv_E)$;

an arrow $[f]\colon E \to F$ from the equilogical space E to the equilogical space F is an equivalence class of continuous equivariant functions with respect to the equivalence relation $\equiv_{F^E}$. We may refer to such an arrow as an ***equivariant map*** in $\mathcal{Equ}$, often confusing a map with one of its representatives;

composition of equivariant maps is defined by composition of a(ny) pair of continuous representatives. Hence, composition is associative and has identities.

The category $\mathcal{Equ}$ is locally cartesian closed (see [CR00]) and it fully extends the category $\mathcal{Top}_0$ of T_0-spaces and continuous functions; the functor

$$\begin{array}{ccc} \mathcal{Top}_0 & \xrightarrow{\quad \mathtt{Y} \quad} & \mathcal{Equ} \\ (|X|, \tau_X) & \longmapsto & (|X|, \tau_X, =) \end{array}$$

is a full embedding and preserves products and all the exponentials which exist in $\mathcal{Top}_0$, see [Sco96].

Note that if $[f]$ is an equivariant map whose target is in the image of the functor $\mathtt{Y}$, then the equivalence class of f is a singleton.

Since the cartesian closed structure of $\mathcal{Equ}$ plays an essential role in what follows, it is useful to recall also Scott's equivalent presentation of $\mathcal{Equ}$ from [Sco96]. It involves algebraic lattices and partial equivalence relations on them. In order to keep the presentation reasonably self-contained, in the following we review some basic concepts from the theory of lattices and universal algebra. We shall refer mainly to [GHK+80].

On a complete lattice $L = (|L|, \leq)$ one can introduce a T_0-topology, called the ***Scott topology*** which consists of those subsets U of $|L|$ such that:

U is ***upward closed***: if $x \in U$ and y is an element of $|L|$ such that $x \leq y$, then $y \in U$;

U is ***inaccessible by directed joins***: for every directed subset $D \subseteq |L|$, if $\bigvee D \in U$, then there exists $d \in D$ such that $d \in U$.

It is easy to check that this collection of sets is closed under arbitrary unions and finite intersections; we will denote that topology with τ_{Sc}. Moreover, given L and M complete lattices, a function $f\colon |L| \to |M|$ is continuous with respect to the Scott topologies on L and M respectively, if and only if f preserves directed joins, *i.e.* for every directed subset $D \subseteq |L|$,

$$f\left(\bigvee_{d\in D} d\right) = \bigvee_{d\in D} f(d).$$

In that case, we shall say that the function $f\colon |L| \to |M|$ is ***Scott-continuous***. In the following we may sometimes confuse a complete lattice L with the topological space $(|L|, \tau_{\mathrm{Sc}})$.

Let L be a complete lattice and $x, y \in |L|$. One says that x is ***way-below*** y, in symbols $x \ll y$, if, for every directed subset $D \subseteq |L|$ such that $y \leq \bigvee D$, there exists $d \in D$ such that $x \leq d$. It is easy to see that the relation $\ll$ is finer than $\leq$, and that $\ll$ is transitive.

An element k of a complete lattice L is ***compact*** if $k \ll k$, *i.e.* for every directed subset $D \subseteq |L|$ such that $k \leq \bigvee D$, there is $d \in D$ such that $k \leq d$. The least element of a complete lattice is compact and a finite join of compact elements is again compact. Note that a compact element k determines a Scott-open subset $k^{\leq} := \left\{ y \in |L| \;\middle|\; k \leq y \right\}$ of L. Denote by $\mathrm{K}(L)$ the subset of the compact elements of L. It is a $\vee$-subsemilattice of L.

A complete lattice L is ***algebraic*** if every element $a \in |L|$ is the join of the compact elements less than or equal to it:

$$a = \bigvee_{\substack{k\in\mathrm{K}(L)\\ k\leq a}} k.$$

Note that the join in the formula above is directed.

The category $\mathcal{AlgLatt}$ of algebraic lattices and Scott-continuous functions is the full subcategory of $\mathcal{Top}_0$ on the algebraic lattices endowed with the Scott topology.

Recall from [Sco76, GHK+80] that an algebraic lattice endowed with the Scott topology is injective with respect to the subspace inclusions in the category $\mathcal{Top}_0$ and that every T_0-space $X = (|X|, \tau_X)$ embeds as a subspace into the algebraic lattice on the powerset $\mathscr{P}(\tau_X)$ ordered by inclusion: the embedding maps a point $x \in |X|$ to its neighbourhood filter $\mathcal{U}_x := \left\{ U \in \tau_X \;\middle|\; x \in U \right\} \in \mathscr{P}(\tau_X)$.

Remark 2.1. Let L and M be algebraic lattices considered as topological spaces with the respective Scott topologies. Let X be a T_0-space and $f\colon L \times X \to M$ a function which is continuous in each variable. Then f is continuous from $L \times X$ into M. Indeed, suppose that $a \in |L|$, $x \in |X|$ and $f(a,x) \in V$ which is Scott-open in M. Since L is algebraic, $a = \bigvee_{\substack{k\in \mathrm{K}(L)\\ k\leq a}} k$. The hypothesis that f is continuous in the first variable ensures that

$$f(a,x) = \bigvee_{\substack{k\in \mathrm{K}(L)\\ k\leq a}} f(k,x).$$

Hence there is a compact $k \leq a$ such that $f(k,x) \in V$ because V is Scott-open. Since f is continuous in the second variable, there is an open neighbourhood U of x such that the image of $\{k\} \times U$ under f is all contained in V. Therefore $f(k^{\leq} \times U) \subseteq V$.

Note that the argument just presented does not extend to arbitrary complete lattices with the Scott topology.

It follows from 2.1 that, given algebraic lattices L and M, the set of Scott-continuous functions from L to M endowed with the compact-open topology is the exponential M^L of the two spaces in $\mathcal{Top}_0$. It is clear that M^L is also a complete lattice, and it is easy to see that it is algebraic and the compact-open topology coincides with the Scott topology. So the embedding $\mathtt{Y}$ restricted to the subcategory of algebraic lattices

$$\begin{array}{ccc} \mathcal{AlgLatt} & \xhookrightarrow{\ \mathtt{I}\ } & \mathcal{Equ} \\ A & \longmapsto & (|A|, \tau_{\mathrm{Sc}}, =) \end{array}$$

preserves products and exponentials.

Remark 2.2. Given algebraic lattices A and B, every order-preserving function $f\colon \mathrm{K}(A) \to B$ has a unique extension to a Scott-continuous function $\tilde{f}\colon A \to B$ mapping

$$a \longmapsto \bigvee_{\substack{k\in \mathrm{K}(A)\\ k\leq a}} f(k).$$

Thus, there is an order isomorphism between the set of Scott-continuous functions from A to B and the order-preserving functions from $\mathrm{K}(A)$ to B.

A complete lattice L is ***continuous*** if every element is the join of the elements way-below it, *i.e.* for every $x \in |L|$ it is

$$x = \bigvee_{\substack{y\in |L|\\ y\ll x}} y.$$

On a continuous lattice the way-below relation is interpolative; in fact, $\ll$ is interpolative on the complete lattice L if and only if L is continuous.

A continuous retract of a continuous lattice is clearly continuous, and every continuous lattice is a retract of an algebraic lattice, *e.g.* for a continuous lattice L, the function

$$\begin{array}{ccc} \mathscr{P}(|L|) & \longrightarrow & \mathscr{P}(|L|) \\ P & \longmapsto & \left\{ y \in |L| \;\middle|\; y \ll \bigvee P \right\} \end{array}$$

is Scott-continuous and the lattice of its fixpoints is (isomorphic to) L, see [Sco76].

So the category $\mathcal{ContLatt}$ of continuous lattices and Scott-continuous functions is (equivalent to) the full subcategory of $\mathcal{Top}_0$ on the injectives with respect to subspace inclusion. It is also equivalent to the idempotent splitting of $\mathcal{AlgLatt}$, so the full embedding $\mathtt{I}$ extends to a full embedding

$$\begin{array}{ccc} \mathcal{ContLatt} & \xhookrightarrow{\mathtt{I}} & \mathcal{Equ} \\ C & \longmapsto & (|C|, \tau_{\mathrm{Sc}}, =) \end{array}$$

which preserves products and exponentials.

We are now in a position to introduce the category $\mathcal{PEqu}$ of ***partial equilogical spaces***.

A ***partial equilogical space*** is a pair $A = (L_A, \approx_A)$, where L_A is an algebraic lattice and $\approx_A$ is a symmetric and transitive relation on $|L_A|$ (not necessarily reflexive). We denote the domain of $\approx_A$ as $D_A := \{a \in |L_A| \mid a \approx_A a\}$.

Given partial equilogical spaces $(L_A, \approx_A)$ and $(L_B, \approx_B)$, for Scott-continuous functions $g, g' \colon L_A \to L_B$, write $g \approx_{B^A} g'$ when

$$\text{for all } a, a' \in |L_A| \text{ such that } a \approx_A a', \text{ it is } g(a) \approx_B g'(a'). \tag{1}$$

For a Scott-continuous function $f \colon L_A \to L_B$ say that it is equivariant from $(L_A, \approx_A)$ to $(L_B, \approx_B)$ when $f \approx_{B^A} f$. So $\approx_{B^A}$ is an equivalence relation on equivariant functions from $(L_A, \approx_A)$ to $(L_B, \approx_B)$. Also, if f is equivariant from $(L_A, \approx_A)$ to $(L_B, \approx_B)$, then it applies D_A into D_B.

The category $\mathcal{PEqu}$ consists of

objects are partial equilogical spaces;

an arrow $[f] \colon A \to B$ from $(L_A, \approx_A)$ to $(L_B, \approx_B)$ is an equivalence class of equivariant functions $f \colon A \to B$ with respect to the equivalence relation $\approx_{B^A}$. We refer to such an arrow as an ***equivariant map*** in $\mathcal{PEqu}$;

composition of equivariant maps is defined by composition of a(ny) pair of continuous representatives.

Hence, composition is associative and has identities.

Remark 2.3. The category $\mathcal{PEqu}$ is the quotient completion of the elementary doctrine $P \colon \mathcal{AlgLatt}^{\mathrm{op}} \longrightarrow \mathcal{InfSL}$ where $P(|L|)$ is the powerset of the underlying set of L and $P(f) := f^{-1}$ for $f \colon L \to M$ a Scott-continuous function, see [MR13, MR15].

It follows from symmetry and transitivity that the partial equivalence relation $\approx_A$ is contained in $D_A \times D_A$ and so reflexive on D_A. That allows to define a functor

$$\begin{array}{ccc}
\mathcal{PEqu} & \xrightarrow{\quad \mathtt{Z} \quad} & \mathcal{Equ} \\
& & \\
(L_A, \approx_A) & \longmapsto & (D_A, \tau_{\text{sub}}, \approx_A) \\
\Big\downarrow [f] & \longmapsto & \Big[f \restriction_{D_A}^{D_B}\Big] \Big\downarrow \\
(L_B, \approx_B) & \longmapsto & (D_B, \tau_{\text{sub}}, \approx_B)
\end{array}$$

where τ_{sub} denotes the appropriate subspace topology.

There is also a functor $\mathtt{W}\colon \mathcal{Equ} \longrightarrow \mathcal{PEqu}$ which exploits the facts, recalled from [Sco76, GHK+80] on p. 123, that the continuous function $x \mapsto \mathcal{U}_x \colon (|X|, \tau_X) \to (\mathscr{P}(\tau_X), \tau_{\text{Sc}})$ is a topological embedding and algebraic lattices are injectives with respect to topological embeddings. The action of W on the objects is

$$\begin{array}{ccc}
\mathcal{Equ} & \xrightarrow{\quad \mathtt{W} \quad} & \mathcal{PEqu} \\
(|E|, \tau_E, \equiv_E) & \longmapsto & (\mathscr{P}(\tau_E), \approx_{\mathscr{P}(\tau_E)})
\end{array}$$

where $\approx_{\mathscr{P}(\tau_E)}$ is the image under $\mathcal{U}_{(-)}$ of the equivalence relation $\equiv_E$; the action on the maps is obtained by injectivity.

Theorem 3.4 in [Sco96] gives the equivalence of categories.

Theorem 2.4. The functors $\mathtt{Z}\colon \mathcal{PEqu} \to \mathcal{Equ}$ and $\mathtt{W}\colon \mathcal{Equ} \to \mathcal{PEqu}$ are an adjoint equivalence.

The following proposition explains how to compute exponentials in $\mathcal{PEqu}$. From this, using the functors Z and W, one derives a construction of exponentials in $\mathcal{Equ}$.

Proposition 2.5. Let A and B be objects in $\mathcal{PEqu}$. Then

(i) their categorical product is
$$A \times B = (L_A \times L_B, \approx_{A\times B})$$
where $(a, b) \approx_{A\times B} (a', b')$ if $a \approx_A a'$ and $b \approx_B b'$, with the obvious projections;

(ii) their exponential is
$$B^A = (L_B^{L_A}, \approx_{B^A})$$
where $L_B^{L_A}$ is the algebraic lattice (ordered pointwise) of the Scott-continuous functions from L_A to L_B introduced in 2.1, $\approx_{B^A}$ is the relation (1), and the evaluation map is that on the algebraic lattices.

Finally, we introduce two subcategories of $\mathcal{PEqu}$ which play a fundamental role in the following, see [FRS17].

Let $\mathcal{REqu}$ be the full subcategory of $\mathcal{PEqu}$ consisting of those pairs $A = (L_A, \equiv_A)$ such that $\equiv_A$ is reflexive, *i.e.* $D_A = |L_A|$. In other words $\equiv_A$ is an equivalence relation on $|L_A|$.

Furthermore, $\mathcal{SEqu}$ is the full subcategory of $\mathcal{PEqu}$ consisting of those pairs $A = (L_A, \sim_A)$, where $\sim_A$ is a ***subreflexive*** relation on $|L_A|$, *i.e.* for all $a, a' \in |L_A|$, if $a \sim_A a'$, then $a = a'$. The category $\mathcal{SEqu}$ is equivalent, under the restriction of the functor $\mathtt{Z}\colon \mathcal{PEqu} \to \mathcal{Equ}$, to the image of the embedding $\mathtt{Y}\colon \mathcal{Top}_0 \hookrightarrow \mathcal{Equ}$.

Recall from [FRS17] the following result.

Proposition 2.6. For S an object in $\mathcal{SEqu}$ and R an object in $\mathcal{REqu}$,

(i) S^R is in $\mathcal{SEqu}$;

(ii) R^S is in $\mathcal{REqu}$.

Remark 2.7. Though the proof of 2.6 is not difficult, it is hard to evaluate its structural meaning. In order to explain what we mean, consider how an object $A = (L_A, \approx_A)$ in $\mathcal{PEqu}$ appears in the following diagram

$$\begin{array}{ccc} (L_A, \Delta_{|L_A|} \cap (D_A \times D_A)) & \xrightarrow{[\mathrm{id}_{L_A}]} & (L_A, \Delta_{|L_A|}) \\ {\scriptstyle [\mathrm{id}_A]}\downarrow & & \\ (L_A, \approx_A) & & \end{array} \tag{2}$$

where $\Delta_{|L_A|}$ denotes the diagonal relation on $|L_A|$. The horizontal map is a subspace inclusion and the vertical map is a coequalizer of the two parallel maps

$$(L_A \times L_A, \Delta_{\approx_A}) \rightrightarrows (L_A, \Delta_{|L_A|} \cap (D_A \times D_A))$$

represented by the two projections.

A partial equilogical space A is in $\mathcal{REqu}$ if and only if the horizontal map in (2) is iso; it is in $\mathcal{SEqu}$ if and only if the vertical map in (2) is iso.

So 2.6(i) is a direct computation using the properties with limits and colimits of an exponential bifunctor. On the other hand, while 2.6(ii) is certainly correct, we failed to find a general justification for it.

From now on, we shall work preferably with partial equilogical spaces. Therefore we shall refer to the category $\mathcal{PEqu}$ rather than the category $\mathcal{Equ}$, as well as its full subcategories $\mathcal{SEqu}$ and $\mathcal{REqu}$. We remark once more that, via the equivalence between $\mathcal{Equ}$ and $\mathcal{PEqu}$, the image of the embedding of $\mathcal{Top}_0$ into $\mathcal{Equ}$ is equivalent to $\mathcal{SEqu}$. We shall show in section 3 that the category of continuous lattices is equivalent to the intersection of $\mathcal{SEqu}$ and $\mathcal{REqu}$.

3 The monad of the double power of Σ

The Sierpinski space Σ is the T_0-space with two points $\bot$ and $\top$ and the only non-trivial open subset is $\{\top\}$. Clearly, Σ is an algebraic lattice, with the order $\bot < \top$ with the Scott topology. So, the pair $(\Sigma, =)$ is a partial equilogical space. For simplicity, in the following, we will write the partial equilogical space $(\Sigma, =)$ simply as Σ.

The self-adjoint functor

$$\mathcal{PEqu} \xrightarrow{\Sigma^{(-)}} \mathcal{PEqu}^{\mathrm{op}}$$

gives rise to a strong monad on $\mathcal{PEqu}$ of the form of those studied in [Tay02a, Tay02b, Vic04, VT04], whose endofunctor $\Sigma^{(\Sigma^{(-)})}$ maps each partial equilogical space E into $\Sigma^{(\Sigma^E)}$—hence the name ***double-power of*** Σ for the monad.

The unit of the monad has components $\eta_E \colon E \to \Sigma^{(\Sigma^E)}$, the exponential adjunct of the composite

$$E \times \Sigma^E \xrightarrow{\langle \pi_2, \pi_1 \rangle} \Sigma^E \times E \xrightarrow{\text{ev}} \Sigma.$$

Since typed λ-calculus can be interpreted in any cartesian closed category, in λ-notation the above map is written

$$\lambda F \colon \Sigma^E . Fx \quad \text{in context } x \colon E.$$

The multiplication component $\mu_E \colon \Sigma^{(\Sigma^{(\Sigma^{(\Sigma^E)})})} \longrightarrow \Sigma^{(\Sigma^E)}$ is the map $\Sigma^{\eta_{\Sigma^E}}$. In λ-notation

$$\lambda F \colon \Sigma^E . G(\lambda U \colon \Sigma^{(\Sigma^E)} . UF) \quad \text{in context } G \colon \Sigma^{(\Sigma^{(\Sigma^{(\Sigma^E)})})}.$$

We shall sometimes adopt the notation of [Tay02a, Tay02b] and write the action Σ^X as $\Sigma\,(X)$, so that $\Sigma^{(\Sigma^X)}$ is written $\Sigma\,(\Sigma\,(X)) = \Sigma^2\,(X)$ and so on. In this way the multiplication above is written $\mu_E \colon \Sigma^4\,(E) \longrightarrow \Sigma^2\,(E)$.

In line with the new notation Σ^2 for the underlying functor of the double-power monad, we shall denote the monad as $\mathbf{\Sigma^2}$ so that the category of the Eilenberg-Moore algebras for it in $\mathcal{PEqu}$ is $\mathcal{PEqu}^{\mathbf{\Sigma^2}}$. A $\mathbf{\Sigma^2}$-algebra is (E, α), where $\alpha \colon \Sigma^2\,(E) \to E$ is a structure map on the partial equilogical space E.

Note that $\Sigma \xrightarrow{\cong} \Sigma^{(\Sigma^0)}$ is the underlying object of the initial $\mathbf{\Sigma^2}$-algebra $(\Sigma, \Sigma^{\eta_1})$. So, for each partial equilogical space E, $(\Sigma^E, \Sigma^{\eta_E})$ is a $\mathbf{\Sigma^2}$-algebra in $\mathcal{PEqu}$ on the power Σ^E of Σ.

Since Σ is both in $\mathcal{REqu}$ and in $\mathcal{SEqu}$, by 2.6 the functor $\Sigma^{(-)} \colon \mathcal{PEqu} \longrightarrow \mathcal{PEqu}^{\text{op}}$ can be restricted and corestricted to the subcategories $\mathcal{REqu}$ and $\mathcal{SEqu}$ in the following way:

$$\mathcal{REqu} \xrightarrow{\Sigma^{(-)}} \mathcal{SEqu}^{\text{op}} \qquad\qquad \mathcal{SEqu} \xrightarrow{\Sigma^{(-)}} \mathcal{REqu}^{\text{op}}$$

Hence, the monad $\mathbf{\Sigma^2}$ gives rise to a monad on $\mathcal{REqu}$ and a monad on $\mathcal{SEqu}$. As usual, we denote the categories of the algebras for the double-power monad of Σ on $\mathcal{REqu}$ and $\mathcal{SEqu}$ with $\mathcal{REqu}^{\mathbf{\Sigma^2}}$ and $\mathcal{SEqu}^{\mathbf{\Sigma^2}}$, respectively.

Since a continuous lattice is a retract of an algebraic lattice, the embedding

$$\mathtt{W} \circ \mathtt{I} \colon \mathcal{ContLatt} \hookrightarrow \mathcal{PEqu}$$

maps into both subcategories $\mathcal{REqu}$ and $\mathcal{SEqu}$.

Lemma 3.1. Let $X = (L_X, \sim_X)$ be an object in $\mathcal{SEqu}$ isomorphic to an object in $\mathcal{REqu}$. Then X is a retract of an algebraic lattice.

Proof. Suppose $X = (L_X, \sim_X)$ with $\sim_X \subseteq \Delta_{|L_X|}$ is isomorphic to an object of $\mathcal{REqu}$; this means that there are an object $A = (L_A, \equiv_A)$ in $\mathcal{REqu}$ and equivariant maps $[f]: X \to A$ and $[g]: A \to X$ such that

$$\begin{array}{ccc} (L_X,\sim_X) & \xrightarrow{\quad\mathrm{id}_{L_X}\quad} & (L_X,\sim_X). \\ {\scriptstyle f}\downarrow \quad \searrow^{[f]} & & \nearrow_{[g]} \\ (L_A,\Delta_{|L_A|}) \xrightarrow[{[\mathrm{id}_{L_A}]}]{} (L_A,\equiv_A) & & \end{array}$$

So X is a retract of the algebraic lattice L_A since $\sim_X \subseteq \Delta_{|L_X|}$. Q.E.D.

Theorem 3.2. The intersection of $\mathcal{REqu}$ and $\mathcal{SEqu}$ is (equivalent to) the image of the embedding $\mathcal{ContLatt} \hookrightarrow \mathcal{PEqu}$.

Proof. It follows from 3.1 since $\mathcal{ContLatt}$ is equivalent to the full subcategory of injectives of $\mathcal{Top}_0$ with respect to subspace inclusion as mentioned on p. 125. Q.E.D.

Hence the functor $\Sigma^{(-)}$ restricts to the category $\mathcal{ContLatt}$ as well as $\mathcal{AlgLatt}$ and we shall also consider the categories of $\mathbf{\Sigma^2}$-algebras in these subcategories.

4 $\mathbf{\Sigma^2}$-algebras and frames

In [FRS17] Theorem 5.5 shows that a $\mathbf{\Sigma^2}$-algebra inherits a unique frame structure in $\mathcal{PEqu}$, induced by the frame structure of Σ. Indeed, by [Dub70], every $\mathbf{\Sigma^2}$-algebra (E, α) can be seen as a $\mathcal{PEqu}$-enriched cotensor-preserving functor

$$\begin{array}{ccc} \left(\mathcal{PEqu}_{\mathbf{\Sigma^2}}\right)^{\mathrm{op}} & \xrightarrow{\quad E^{(-)}\quad} & \mathcal{PEqu} \\ D & \longmapsto & E^D \end{array}$$

Note that $\left(\mathcal{PEqu}_{\mathbf{\Sigma^2}}\right)^{\mathrm{op}}$ is equivalent to the ***theory of*** Σ ***in*** $\mathcal{PEqu}$, *i.e.* $\mathcal{Th}(\Sigma)$ is the category whose objects are the objects of $\mathcal{PEqu}$ and an arrow $f: F \to G$ is an equivariant map $f: \Sigma^F \to \Sigma^G$; composition and identities of $\mathcal{Th}(\Sigma)$ are as in $\mathcal{PEqu}$. Thus, applying the functor $E^{(-)}$ to the distributive lattice structure of Σ, given by the Scott-continuous functions

$$\wedge: \Sigma^2 \to \Sigma \qquad \vee: \Sigma^2 \to \Sigma,$$

we obtain distributive lattice operations on the underlying object E of the $\mathbf{\Sigma^2}$-algebra (E, α).

Remark 4.1. We should remind the reader that the notation $E^{(-)}$ is only suggestive, the action on the arrows is *not* just by pre-composition and uses the structure map α, see [FRS17]. Indeed, if $f: C \to D$ is an arrow in $\mathcal{Th}(\Sigma)$, then it is an equivariant map $f: \Sigma^C \to \Sigma^D$ and E^f is represented by the equivariant function

$$E^C \xrightarrow{(\eta_E)^C} (\Sigma^{(\Sigma^E)})^C \cong (\Sigma^C)^{(\Sigma^E)} \xrightarrow{f^{(\Sigma^E)}} (\Sigma^D)^{(\Sigma^E)} \cong (\Sigma^{(\Sigma^E)})^D \xrightarrow{\alpha^D} E^D$$

In particular, the order determined on E is given as follows: let $\mathtt{2} = \mathtt{1} + \mathtt{1}$ be the discrete equilogical space on the set $\{0, 1\}$ and $\binom{\perp}{\top}\colon \mathtt{1} + \mathtt{1} \to \Sigma$ the function which maps 0 to $\perp$ and 1 to $\top$. Then $\Sigma^{\binom{\perp}{\top}}\colon \Sigma^\Sigma \to \Sigma^{\mathtt{1}+\mathtt{1}}$ is monic and isomorphic to the order relation on Σ. An easy diagram chase shows that $E^{\Sigma^{\binom{\perp}{\top}}}$ is represented by the equivariant function

$$\begin{array}{ccc} E^\Sigma & \longrightarrow & E^2 \cong E \times E \\ f & \longmapsto & (f(\perp), f(\top)) \end{array}$$

which is indipendent of the structure α. Hence the internal distributive lattice determined in $\mathcal{PEqu}$ on E depends only on the existence of a structure map α on E that turns it into a $\mathbf{\Sigma^2}$-algebra. For this reason we shall denote the maps $E^\wedge\colon E^2 \to E$ and $E^\vee\colon E^2 \to E$, obtained by applying $E^{(-)}$ to the maps

$$\wedge\colon \Sigma^2 \to \Sigma \qquad \vee\colon \Sigma^2 \to \Sigma,$$

simply as $\underset{E}{\wedge}\colon E^2 \to E$ and $\overset{E}{\vee}\colon E^2 \to E$.

Moreover, for every set I, seen as a discrete topological space, the join $\bigvee^I\colon \Sigma^I \to \Sigma$ is Scott-continuous, so they induce (arbitrary) join operations

$$\overset{E}{\bigvee}{}^I\colon E^I \to E$$

which make E an internal frame in $\mathcal{PEqu}$. If it causes no confusion, we omit the index I.

The following is an explicit description of the induced lattice operations in terms of representatives of the equivariant maps of partial equilogical spaces:

$$\underset{E}{\wedge}\colon (e_1, e_2) \mapsto \alpha\left(\eta_E(e_1) \wedge \eta_E(e_2)\right)$$

$$\overset{E}{\bigvee}{}^I\colon (e_i)_{i\in I} \mapsto \alpha\left(\bigvee\nolimits_{i\in I} \eta_E(e_i)\right)$$

where $\wedge$ and $\bigvee$ which appear on the right-hand side in the definition above are the pointwise finite meet and arbitrary join of continuous functions.

Again, in terms of representatives, if $h\colon (E, \alpha) \to (D, \beta)$ is a $\mathbf{\Sigma^2}$-homomorphism, then h is a frame homomorphism up to the partial equivalence relation $\approx_D$, in the sense that

$$h(e_1 \underset{E}{\wedge} e_2) \approx_D h(e_1) \underset{D}{\wedge} h(e_2) \qquad \text{for } e_1, e_2 \in D_E$$

$$h\left(\overset{E}{\bigvee} e_i\right) \approx_D \overset{D}{\bigvee} h(e_i) \qquad \text{for } (e_i)_{i\in I} \in (D_E)^I.$$

As a direct consequence, the global section functor $\Gamma\colon \mathcal{PEqu} \to \mathcal{Set}$ extends to a faithful functor

$$\begin{array}{ccc} \mathcal{PEqu}^{\mathbf{\Sigma^2}} & \xrightarrow{\ \Gamma\ } & \mathcal{Frm} \\ (E, \alpha) & \longmapsto & D_E/\approx_E \end{array}$$

where $\mathcal{Frm}$ is the category of frames and frame homomorphisms. We denote the frame operations on $\Gamma(E, \alpha) = D_E/\approx_E$ with $\underset{E}{\sqcap}$ and $\overset{E}{\bigsqcup}$.

Remark 4.2. Clearly, the frame structure of $\Gamma(E, \alpha)$ is unique and depends only on the existence of a $\mathbf{\Sigma^2}$-structure on the object E.

Consider the particular case of $\mathbf{\Sigma^2}$-algebras in $\mathcal{AlgLatt}$ and suppose that (A, α) is in $\mathcal{AlgLatt}^{\mathbf{\Sigma^2}}$, *i.e.* $A = (L_A, =)$. Then the frame structure determined on A coincides with that given by the complete order on $L_A = (|L_A|, \wedge, \bigvee)$ since the mono

$$A^{\Sigma^{\left(\frac{\bot}{\top}\right)}} \colon A^{\Sigma} \rightarrowtail A^2$$

is (isomorphic) to the order relation of the algebraic lattice A. Therefore, every $\mathbf{\Sigma^2}$-homomorphism $h\colon (A, \alpha) \to (B, \beta)$ preserves arbitrary joins and finite meets. Since each structure map $\alpha\colon \Sigma^{(\Sigma^A)} \to A$ is a $\mathbf{\Sigma^2}$-homomorphism from $(\Sigma^{(\Sigma^A)}, \mu_A)$ to (A, α), it preserves finite meets and arbitrary joins. This allows us to prove the following.

Lemma 4.3. Let (A, α) and (B, β) be objects in $\mathcal{AlgLatt}^{\mathbf{\Sigma^2}}$. If $h\colon L_A \to L_B$ is a frame homomorphism, then it is a $\mathbf{\Sigma^2}$-homomorphism from (A, α) to (B, β).

Proof. We have to prove that, given $h\colon L_A \to L_B$ a frame homomorphism, the following diagram is commutative:

$$\begin{array}{ccc} \Sigma^{(\Sigma^A)} & \xrightarrow{\Sigma^{(\Sigma^h)}} & \Sigma^{(\Sigma^B)} \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ A & \xrightarrow[h]{} & B \end{array}$$

Since the lattices involved are algebraic and all the maps in the diagram are Scott-continuous, it is sufficient to prove that the diagram commutes on the compact elements of $\Sigma^{(\Sigma^{L_A})}$; they are finite joins of step functions, so they are of the form

$$\bigvee_{i=1}^{n} \bigwedge_{j=1}^{m} \widehat{k_{ij}},$$

for appropriate compact elements k_{ij} of L_A. The function $\widehat{k_{ij}} = \eta_A(k_{ij})$ maps $f \in |\Sigma^{L_A}|$ into the function $f(k_{ij})$. Thus, computing the two paths on a step function, we obtain

$$\begin{array}{ccc} \bigvee_{i=1}^{n} \bigwedge_{j=1}^{m} \widehat{k_{ij}} & \xmapsto{\Sigma^{(\Sigma^h)}} & \bigvee_{i=1}^{n} \bigwedge_{j=1}^{m} \widehat{h(k_{ij})} \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ \bigvee_{i=1}^{n} \bigwedge_{j=1}^{m} k_{ij} & \xmapsto[h]{} & h(\bigvee_{i=1}^{n} \bigwedge_{j=1}^{m} k_{ij}) = \bigvee_{i=1}^{n} \bigwedge_{j=1}^{m} h(k_{ij}) \end{array}$$

which completes the proof. Q.E.D.

We shall extend the previous results to $\mathbf{\Sigma^2}$-algebras in $\mathcal{REqu}$. First we need a result about the $\mathbf{\Sigma^2}$-algebras which are powers of Σ.

Theorem 4.4. Let $E = (L_E, \approx_E)$ be a partial equilogical space. For all $f_1, f_2 \in D_{\Sigma^E}$ and $\{f_i\}_{i\in I} \subseteq D_{\Sigma^E}$,

$$f_1 \underset{\Sigma^E}{\wedge} f_2 \approx_{\Sigma^E} f_1 \wedge f_2 \qquad \overset{\Sigma^E}{\bigvee} f_i \approx_{\Sigma^E} \bigvee f_i,$$

where the operations $\wedge$ and $\bigvee$ which appear on the right-hand side of the identities are the pointwise meet and join of the algebraic lattice Σ^{L_E}.

Proof. Write L for the partial equilogical space $(L_E, =_{L_E})$—in other words, L is the algebraic lattice L_E *seen as* a partial equilogical space. Assume first that E is in $\mathcal{SE}qu$. Note that $\left[\mathrm{id}_{|L_E|}\right] : E \rightarrowtail L$. Since the following commutative diagram

$$\begin{array}{ccc}
\Sigma^2\left(\Sigma^L\right) & \xrightarrow{\Sigma^{\eta_L}} & \Sigma^L \\
{\scriptstyle \Sigma^2\left(\Sigma^{\left[\mathrm{id}_{|L_E|}\right]}\right)}\downarrow & & \downarrow{\scriptstyle \Sigma^{\left[\mathrm{id}_{|L_E|}\right]}} \\
\Sigma^2\left(\Sigma^E\right) & \xrightarrow{\Sigma^{\eta_E}} & \Sigma^E
\end{array}$$

and the vertical maps are surjections by 2.6, given $f_1, f_2 \in |\Sigma^{L_E}|$,

$$f_1 \underset{\Sigma^E}{\wedge} f_2 = \Sigma^{\eta_E}(\eta_{\Sigma^E}(f_1) \wedge \eta_{\Sigma^E}(f_2)) \equiv_{\Sigma^E} \Sigma^{\eta_L}(\eta_{\Sigma^L}(f_1) \wedge \eta_{\Sigma^L}(f_2)) = f_1 \wedge f_2$$

where $\wedge$ is the pointwise meet of the algebraic lattice Σ^{L_E}.

The proof is similar for $\overset{\Sigma^E}{\bigvee}$.

For the general case of E a partial equilogical space, write X for the partial equilogical space $(L_E, \Delta_{|L_E|} \cap (D_E \times D_E))$, see 2.7. Note that $\left[\mathrm{id}_{|L_E|}\right] : X \longrightarrow E$ and consider the commutative diagram

$$\begin{array}{ccc}
\Sigma^2\left(\Sigma^E\right) & \xrightarrow{\Sigma^{\eta_E}} & \Sigma^E \\
{\scriptstyle \Sigma^2\left(\Sigma^{\left[\mathrm{id}_{|L_E|}\right]}\right)}\downarrow & & \downarrow{\scriptstyle \Sigma^{\left[\mathrm{id}_{|L_E|}\right]}} \\
\Sigma^2\left(\Sigma^X\right) & \xrightarrow{\Sigma^{\eta_X}} & \Sigma^X
\end{array}$$

where the vertical maps are monic by 2.6. The result follows immediately. Q.E.D.

Consider now (A, α) and (B, β) in $\mathcal{RE}qu^{\mathbf{\Sigma^2}}$. If $h: (A, \alpha) \to (B, \beta)$ is a $\mathbf{\Sigma^2}$-homomorphism, then h is an internal frame homomorphism from $(A, \wedge_A, \bigvee^A)$ to $(B, \wedge_B, \bigvee^B)$ in $\mathcal{RE}qu$. So, up to the equivalence relation $\equiv_B$, for all $a_1, a_2 \in |L_A|$ and for all $\{a_i\}_{i\in I} \subseteq |L_A|$

$$h\left(a_1 \wedge_A a_2\right) \equiv_B h(a_1) \wedge_B h(a_2) \qquad \text{and} \qquad h\left(\bigvee^A a_i\right) \equiv_B \bigvee^B h(a_i).$$

Since each structure map $\alpha: \Sigma^{(\Sigma^A)} \to A$ is a $\mathbf{\Sigma^2}$-homomorphism, it is a frame homomorphism from $\Sigma^{(\Sigma^A)}$ to A in $\mathcal{RE}qu$. By Theorem 4.4, for all $G_1, G_2 \in |\Sigma^{(\Sigma^{L_A})}|$ and for all $\{G_i\}_{i\in I} \subseteq |\Sigma^{(\Sigma^{L_A})}|$,

$$\alpha(G_1 \wedge G_2) \equiv_A \alpha(G_1) \wedge_A \alpha(G_2) \qquad \text{and} \qquad \alpha\left(\bigvee G_i\right) \equiv_A \bigvee^A \alpha(G_i).$$

Theorem 4.5. Let (A,α) and (B,β) be objects in $\mathcal{REqu}^{\boldsymbol{\Sigma^2}}$. If h is an equivariant map from A to B which, in addition, is an internal frame homomorphism from $(A, \bigwedge_{A}, \bigvee^{A})$ to $(B, \bigwedge_{B}, \bigvee^{B})$ in $\mathcal{REqu}$, then it is a $\boldsymbol{\Sigma^2}$-homomorphism from (A,α) to (B,β).

Proof. In order to prove that h is a $\boldsymbol{\Sigma^2}$-homomorphism, it is sufficient to prove that the following diagram commutes

$$\begin{array}{ccc} \Sigma^{(\Sigma^A)} & \xrightarrow{\Sigma^{(\Sigma^h)}} & \Sigma^{(\Sigma^B)} \\ \alpha\downarrow & & \downarrow\beta \\ A & \xrightarrow[h]{} & B \end{array}$$

Each function G in $\Sigma^{(\Sigma^{L_A})}$ is the directed join of the compact elements below it. Since each compact element in $\Sigma^{(\Sigma^{L_A})}$ is of the form $\bigvee_{i=1}^{n}\bigwedge_{j=1}^{m}\widehat{k_{ij}}$ where for all i and j, k_{ij} is a compact element of L_A, G is of the form $\bigvee\wedge\widehat{k}$, where the meet in the formula is finite. Therefore,

$$\begin{array}{ccc} \bigvee\wedge\widehat{k} & \xmapsto{\Sigma^{(\Sigma^h)}} & \bigvee\wedge\widehat{h(k)} \\ \alpha\big\downarrow & & \big\downarrow\beta \\ \bigvee^{A}\bigwedge_{A} k & \xmapsto[h]{} & h(\bigvee^{A}\bigwedge_{A} k)\equiv_B \bigvee^{B}\bigwedge_{B} h(k) \end{array}$$

Q.E.D.

Remark 4.6. The previous result does not extend directly to the general case of $\boldsymbol{\Sigma^2}$-algebras in $\mathcal{PEqu}$ because, for a compact element k of L_A, the value $h(k)$ need not be in the domain of the partial equivalence relation $\approx_B$. Furthermore, if (A,α) is a $\boldsymbol{\Sigma^2}$-algebra in $\mathcal{PEqu}$ and $G \in D_{\Sigma^{(\Sigma^A)}}$, then $G = \bigvee\wedge\widehat{k}$ for some appropriate $k \in \mathrm{K}\left(\Sigma^{(\Sigma^A)}\right)$, but we do not know if every $\widehat{k}$ is in $D_{\Sigma^{(\Sigma^A)}}$. So, we cannot conclude that $\alpha(\bigvee\wedge\widehat{k}) \equiv_A \bigvee^{A}\bigwedge_{A} k$.

Theorem 4.7. Let (A,α) be a $\boldsymbol{\Sigma^2}$-algebra in $\mathcal{REqu}$. Then the canonical surjection

$$q_A\colon L_A \to (|L_A|/\equiv_A, \sqcup^{A})$$

preserves directed joins.

Proof. Let $(a_d)_{d\in D}$ be a directed family in $L_A = (|L_A|, \wedge, \vee)$. Then

$$q_A\left(\bigvee a_d\right) = \left[\bigvee a_d\right] = \left[\alpha\left(\widehat{\bigvee a_d}\right)\right] = \left[\alpha\left(\bigvee \widehat{a_d}\right)\right] = \left[\bigvee^{A} a_d\right] = \sqcup^{A} q_A(a_d),$$

where the third equality follows from the fact that η_A preserves directed joins. Q.E.D.

We conclude this section proving uniqueness of a structure of $\boldsymbol{\Sigma^2}$-algebra on particular partial equilogical spaces A.

Theorem 4.8. Let A be an object of $\mathcal{REqu}$, and let (A, α_1) and (A, α_2) be $\mathbf{\Sigma^2}$-algebras. Then $\alpha_1 = \alpha_2$.

Proof. Suppose $A = (L_A, \equiv_A)$ is an object of $\mathcal{REqu}$ and $\alpha_1, \alpha_2 \colon \Sigma^{(\Sigma^A)} \to A$ are structure maps on A. Since they are equivariant maps, in order to prove that they coincide, it is sufficient to show that for every $G \in |\Sigma^{(\Sigma^{L_A})}|$, $\alpha_1(G) \equiv_A \alpha_2(G)$. Since $\Sigma^{(\Sigma^{L_A})}$ is an algebraic lattice, each G is an arbitrary join of finite meets of functions of the form $\widehat{k}$, where k is a compact element of L_A. The fact that α_1 and α_2 are frame homomorphisms from $\Sigma^{(\Sigma^A)}$ to A in $\mathcal{REqu}$ implies that

$$\alpha_1(G) = \alpha_1\left(\bigvee \wedge \widehat{k}\right) \equiv_A \overset{A}{\bigvee} \overset{A}{\wedge}\, \alpha_1\left(\widehat{k}\right) \equiv_A \overset{A}{\bigvee} \overset{A}{\wedge}\, k \equiv_A \alpha_2\left(\bigvee \wedge \widehat{k}\right) = \alpha_2(G). \qquad \text{Q.E.D.}$$

By 3.2, the previous result applies directly to the categories $\mathcal{AlgLatt}^{\mathbf{\Sigma^2}}$ and $\mathcal{ContLatt}^{\mathbf{\Sigma^2}}$.

Corollary 4.9. Let A be an object of $\mathcal{AlgLatt}$, and let (A, α_1) and (A, α_2) be $\mathbf{\Sigma^2}$-algebras. Then $\alpha_1 = \alpha_2$.

Corollary 4.10. Let A be an object of $\mathcal{ContLatt}$, and let (A, α_1) and (A, α_2) be $\mathbf{\Sigma^2}$-algebras. Then $\alpha_1 = \alpha_2$.

5 A characterization for $\mathcal{ContLatt}^{\mathbf{\Sigma^2}}$ and $\mathcal{AlgLatt}^{\mathbf{\Sigma^2}}$

In this section, we show a characterization for the categories of $\mathbf{\Sigma^2}$-algebras in $\mathcal{ContLatt}$ and in $\mathcal{AlgLatt}$. In the following, we denote with $\mathcal{ContFrm}$ the category of continuous frames and frame homomorphisms, and with $\mathcal{AlgFrm}$ the category of algebraic frames and frame homomorphisms.

Theorem 5.1. The categories $\mathcal{ContLatt}^{\mathbf{\Sigma^2}}$ and $\mathcal{ContFrm}$ are equivalent.

Proof. One of the functors involved in the equivalence is the restriction to $\mathcal{ContLatt}^{\mathbf{\Sigma^2}}$ of the global section functor

$$\mathcal{PEqu}^{\mathbf{\Sigma^2}} \xrightarrow{\ \Gamma\ } \mathcal{Frm}$$

For every (C, α) in $\mathcal{ContLatt}^{\mathbf{\Sigma^2}}$, the continuous lattice $\Gamma(C, \alpha)$ is a frame and the same argument to that for algebraic lattices proves that the frame structure on C given by the $\mathbf{\Sigma^2}$-structure coincides with that given by the order on the continuous lattice structure of $\Gamma(C, \alpha)$. In other words, $\Gamma(C, \alpha)$ is a continuous frame, and Γ maps $\mathcal{ContLatt}^{\mathbf{\Sigma^2}}$ into $\mathcal{ContFrm}$.
As for the other functor, we shall employ the construction of the space of points of a frame from [GHK+80, Joh82]. For F a frame, consider the sober topological space $\mathrm{pt}(F)$: its points are the frame homomorphisms $p \colon F \to \Sigma$; its topology consists of the sets $\mathcal{O}(a)$ for $a \in |F|$ where a frame homomorphism $p \colon F \to \Sigma$ is in $\mathcal{O}(a)$ if $p(a) = \top$. It is easy to check that these are closed under finite intersections and arbitrary unions and that

$$\begin{array}{ccc} F & \xrightarrow{\ \mathcal{O}\ } & \mathrm{pt}(F) \\ a & \longmapsto & \mathcal{O}(a) \end{array}$$

is a frame homomorphism. Also the assignment $F \mapsto \mathrm{pt}(F)$ easily extends functorially, mapping a frame homomorphism $f\colon F \to G$ to precomposition with f

$$\begin{array}{ccc} \mathrm{pt}(G) & \xrightarrow{\mathrm{pt}(f)} & \mathrm{pt}(F) \\ p & \longmapsto & p \circ f. \end{array}$$

It is also well-known, see *loc.cit.*, that, when F is a continuous frame, the space $\mathrm{pt}(F)$ is locally compact and $\mathcal{O}$ is an isomorphism. So $\mathrm{pt}(F)$ is an exponentiable topological space and $\Sigma^{\mathrm{pt}(F)}$ is in $\mathcal{ContLatt}$ and a $\mathbf{\Sigma^2}$-algebra. So consider the functor

$$\begin{array}{ccc} \mathcal{ContFrm} & \xrightarrow{\Sigma^{\mathrm{pt}}} & \mathcal{ContLatt}^{\mathbf{\Sigma^2}} \\ F & \longmapsto & \left(\Sigma^{(\mathrm{pt}(F))}, \Sigma^{\eta_{\mathrm{pt}(F)}}\right) \\ \downarrow f & \longmapsto & \Sigma^{\mathrm{pt}(f)} \downarrow \\ G & \longmapsto & \left(\Sigma^{(\mathrm{pt}(G))}, \Sigma^{\eta_{\mathrm{pt}(G)}}\right) \end{array}$$

Now suppose that F is a continuous frame. Then

$$\Gamma(\Sigma^{\mathrm{pt}(F)}) \cong F.$$

If (C, α) is in $\mathcal{ContLatt}^{\mathbf{\Sigma^2}}$, then $\Sigma^{\mathrm{pt}(\Gamma(C,\alpha))} \cong C$ as continuous lattices, hence as partial equilogical spaces. By 4.10, they are isomorphic as $\mathbf{\Sigma^2}$-algebras. Q.E.D.

Theorem 5.2. The categories $\mathcal{AlgFrm}$ and $\mathcal{AlgLatt}^{\mathbf{\Sigma^2}}$ are equivalent

Proof. Since an algebraic lattice is continuous and Σ is algebraic, the functors involved in the proof of Theorem 5.1 can be restricted and corestricted to the categories $\mathcal{AlgFrm}$ and $\mathcal{AlgLatt}^{\mathbf{\Sigma^2}}$

$$\mathcal{AlgFrm} \underset{\Gamma}{\overset{\Sigma^{\mathrm{pt}}}{\rightleftarrows}} \mathcal{AlgLatt}^{\mathbf{\Sigma^2}}$$

Q.E.D.

6 $\mathbf{\Sigma^2}$-algebras in $\mathcal{Top}_0$

Our final aim is to investigate the category $\mathcal{Top}_0^{\mathbf{\Sigma^2}}$ of $\mathbf{\Sigma^2}$-algebras in $\mathcal{Top}_0$. We identified $\mathcal{Top}_0$ with the full subcategory $\mathcal{SEqu}$ of $\mathcal{PEqu}$ consisting of those partial equilogical spaces whose relation is contained in the diagonal. As noted in section 3 using subreflexive partial equivalence relations, although, for a T_0-space X, Σ^X need not be a topological space, $\Sigma^{(\Sigma^X)}$ is always a topological space and the double power of Σ gives rise to a monad on $\mathcal{Top}_0$.

We do not see if the arguments that prove the uniqueness of the structure map in the cases of the categories $\mathcal{AlgLatt}$, $\mathcal{ContLatt}$ and $\mathcal{REqu}$ can be applied to $\mathcal{SEqu}$. Indeed, if α is a structure map on the object A of $\mathcal{SEqu}$, we only know that α is a frame homomorphism on the elements of $D_{\Sigma^{(\Sigma^A)}}$. But, for $G \in D_{\Sigma^{(\Sigma^A)}}$, it is $G = \bigvee \left\{ c \in \mathrm{K}\left(\Sigma^{(\Sigma^{L_A})}\right) \mid c \leq G \right\}$. Alas, the compact elements below G need not belong to $D_{\Sigma^{(\Sigma^A)}}$.

In order to outline some properties of $\mathbf{\Sigma^2}$-algebras in $\mathcal{Top}_0$, we move back to $\mathcal{Equ}$ and compute in $\mathcal{Equ}$ the exponential $\Sigma^{(\Sigma^{(|X|,\tau_X,=)})}$ for $X = (|X|, \tau_X)$ a T_0-space. Thus,

$$\Sigma^{(|X|,\tau_X,=)} = (\Sigma^{\mathscr{P}(\tau_X)}, \tau_{\mathrm{Sc}}, \equiv_{\Sigma^X})$$

where

- $\Sigma^{\mathscr{P}(\tau_X)}$ is the algebraic lattice of Scott-continuous functions from $\mathscr{P}(\tau_X)$ to Σ;
- for Scott-continuous functions $f, g \colon \mathscr{P}(\tau_X) \to \Sigma$, $f \equiv_{\Sigma^X} g$ if for every $x \in |X|$, one has that $f(\mathcal{U}_x) = g(\mathcal{U}_x)$.

We next compute

$$\Sigma^{(\Sigma^{(|X|,\tau_X,=)})} = (|\Sigma^{(\Sigma^X)}|, \tau_{\mathrm{sub}}, =)$$

where

- $|\Sigma^{(\Sigma^X)}|$ is the set of Scott-continuous functions $G\colon \Sigma^{\mathscr{P}(\tau_X)} \to \Sigma$ such that, for all $f, g \in \Sigma^{\mathscr{P}(\tau_X)}$ with $f \equiv_{\Sigma^X} g$, it is $G(f) = G(g)$;
- τ_{sub} is the subspace topology with respect to the inclusion $|\Sigma^{(\Sigma^X)}| \subseteq \Sigma^{(\Sigma^{\mathscr{P}(\tau_X)})}$ into the algebraic lattice.

Remark 6.1. $|\Sigma^{(\Sigma^X)}|$ is a subframe of $\Sigma^{(\Sigma^{\mathscr{P}(\tau_X)})}$. Suppose that $\{G_i\}_{i\in I} \subseteq |\Sigma^{(\Sigma^X)}|$ and take $f, g \in \Sigma^{\mathscr{P}(\tau_X)}$ such that $f \equiv_{\Sigma^X} g$. Then, for all $i \in I$, $G_i(f) = G_i(g)$. Since joins are computed pointwise, $\bigvee_{i\in I} G_i(f) = \bigvee_{i\in I} G_i(g)$; so $\bigvee_{i\in I} G_i \in |\Sigma^{(\Sigma^X)}|$. Similarly for finite meets.

Lemma 6.2. The identity function $\mathrm{id}_{|\Sigma^{(\Sigma^X)}|}\colon (|\Sigma^{(\Sigma^X)}|, \tau_{\mathrm{Sc}}) \to (|\Sigma^{(\Sigma^X)}|, \tau_{\mathrm{sub}})$ is continuous.

Proof. Let $V \subseteq |\Sigma^{(\Sigma^X)}|$ be an open subset with respect to the subspace topology τ_{sub}. So there is a Scott-open set $U \subseteq \Sigma^{(\Sigma^{\mathscr{P}(\tau_X)})}$ such that $V = |\Sigma^{(\Sigma^X)}| \cap U$. Let $F \in V$ and $G \in |\Sigma^{(\Sigma^X)}|$ be such that $F \leq G$. Since $F \in U$ and U is upward closed, $G \in U$. Therefore, $G \in V$. So, V is upward closed in $|\Sigma^{(\Sigma^X)}|$. Suppose now that $\{G_d\}_{d\in D}$ is a directed family of functions in $|\Sigma^{(\Sigma^X)}|$ and suppose that $\bigvee_{d\in D} G_d \in V$. Then $\bigvee_{d\in D} G_d \in U$. Since U is inaccessible by directed joins, there exists $d \in D$ such that $G_d \in U$ and, consequently, $G_d \in V$. So, V is inaccessible by directed joins and it is a Scott-open subset of $|\Sigma^{(\Sigma^X)}|$. Q.E.D.

Theorem 6.3. The topological space $\Sigma^{(\Sigma^X)} = (|\Sigma^{(\Sigma^X)}|, \tau_{\mathrm{sub}})$ is compact and connected.

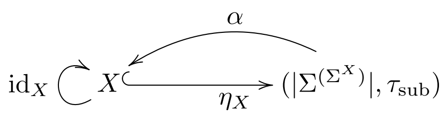
Proof. Let $\{U_i\}_{i\in I} \subseteq |\Sigma^{(\Sigma^X)}|$, open with respect to τ_{sub}, and suppose that $|\Sigma^{(\Sigma^X)}| = \bigcup_{i\in I} U_i$. Since $\text{const}_\perp \in |\Sigma^{(\Sigma^X)}|$, there exists $j \in I$ such that $\text{const}_\perp \in U_j$. But, for all $i \in I$, U_i is upward closed, so $|\Sigma^{(\Sigma^X)}| \subseteq U_j$ and $(|\Sigma^{(\Sigma^X)}|, \tau_{\text{sub}})$ is compact.
Suppose now that there exists a disconnection for $(|\Sigma^{(\Sigma^X)}|, \tau_{\text{sub}})$, namely there are $U_1, U_2 \subseteq |\Sigma^{(\Sigma^X)}|$ in τ_{sub} such that

$$|\Sigma^{(\Sigma^X)}| = U_1 \cup U_2 \qquad U_1 \cap U_2 = \varnothing \qquad U_1 \neq \varnothing \neq U_2.$$

Say $\text{const}_\perp$ belongs to U_1; then $|\Sigma^{(\Sigma^X)}| = U_1$. Then $U_1 \cap U_2 \neq \varnothing$ which is a contradiction. Q.E.D.

Corollary 6.4. If (X, α) is a $\mathbf{\Sigma^2}$-algebra in $\mathcal{T}op_0$, then X is a compact, connected topological space.

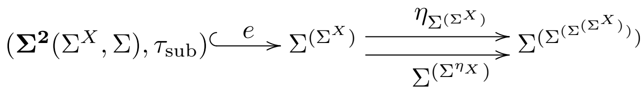
Proof. For a $\mathbf{\Sigma^2}$-algebra (X, α) in $\mathcal{T}op_0$, the following diagram commutes



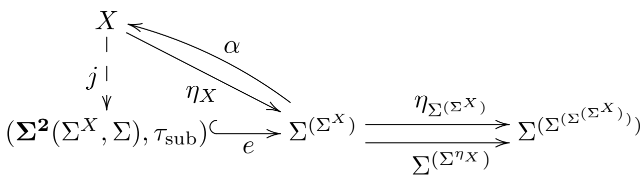
Therefore, by 6.3, X is the image through α of a compact, connected space, so it is a compact, connected space. Q.E.D.

Theorem 6.5. If (X, α) is a $\mathbf{\Sigma^2}$-algebra in $\mathcal{T}op_0$, then X is a sober space.

Proof. For the space of the proof, we denote with $\mathbf{\Sigma^2}(\Sigma^X, \Sigma)$ the set of $\mathbf{\Sigma^2}$-homomorphisms from Σ^X to Σ. Recall from [BR14] that



is an equalizer in $\mathcal{T}op_0$, where τ_{sub} is the subspace topology with respect to the inclusion e. Therefore, we have



We shall show that the functions j and $\alpha \circ e$ are inverse of each other, hence that X and $(\mathbf{\Sigma^2}(\Sigma^X, \Sigma), \tau_{\text{sub}})$ are homeomorphic. By the properties of structure map, $(\alpha \circ e) \circ j = \alpha \circ \eta_X = \text{id}_X$. To prove the other identity, we shall compose it with e and show that $\eta_X \circ \alpha \circ e = e$. Consider $f \in \mathbf{\Sigma^2}(\Sigma^X, \Sigma)$; so, for all $g \in |\Sigma^{(\Sigma^{(\Sigma^X)})}|$, it is $g(f) = f(g \circ \eta_X)$. We have to prove that $\eta_X(\alpha(f))(t) = f(t)$ for all $t \in |\Sigma^X|$. Suppose $t \in |\Sigma^X|$, thus $t \circ \alpha \in |\Sigma^{(\Sigma^{(\Sigma^X)})}|$. Therefore

$$\eta_X(\alpha(f))(t) = t(\alpha(f)) = (t \circ \alpha)(f) = f((t \circ \alpha) \circ \eta_X) = f(t \circ \text{id}_X) = f(t),$$

as required. By [BR14] $(\mathbf{\Sigma^2}(\Sigma^{(|X|, \tau_X, =)}, \Sigma), \tau_{\text{sub}})$ is sober, so also X is a sober space. Q.E.D.

Proposition 6.6. There are $\mathbf{\Sigma^2}$-algebras (X, α) in $\mathcal{Top}_0$ such that X is not an exponentiable topological space.

Proof. We shall show that if X is a non-exponentiable topological space, then $\Sigma^{(\Sigma^X)}$ is a $\mathbf{\Sigma^2}$-algebra in $\mathcal{Top}_0$ which is not exponentiable. Indeed, if $\Sigma^{(\Sigma^X)}$ is exponentiable, then $\Sigma^{(\Sigma^{(\Sigma^X)})}$ is a topological space and, in addition, it is injective because Σ is injective with respect to subspace inclusions and, if Y is an exponentiable T_0-space, then the functor $(-)^Y$ preserves injectives. But Σ^X is a retract in $\mathcal{Equ}$ of $\Sigma^{(\Sigma^{(\Sigma^X)})}$, as

$$\mathrm{id}_{\Sigma^X} \circlearrowright \Sigma^X \underset{\eta_{\Sigma^X}}{\hookrightarrow} \Sigma^{(\Sigma^{(\Sigma^X)})}, \qquad \Sigma^{\eta_X} : \Sigma^{(\Sigma^{(\Sigma^X)})} \to \Sigma^X.$$

Therefore, Σ^X is an injective topological space and so X is exponentiable. Q.E.D.

Remark 6.7. If X is sober, then X is exponentiable if and only if X is locally compact. So, the previous proposition equivalently states that there are $\mathbf{\Sigma^2}$-algebras (X, α) in $\mathcal{Top}_0$ such that X is not locally compact.

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